

Fiber product of prestacks

$\mathcal{X}, \mathcal{Y}', \mathcal{Y}$ prestacks over a category $\mathcal{C}$ w/ morphisms $f: \mathcal{X} \rightarrow \mathcal{Y}$, $g: \mathcal{Y}' \rightarrow \mathcal{Y}$.

We define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{C}$ as follows:

- Objects: (x, y', h) s.t. $x \in \text{Obj}(\mathcal{X})$ & $y' \in \text{Obj}(\mathcal{Y}')$ over some object $S \in \text{Obj}(\mathcal{C})$ and $h: f(x) \xrightarrow{\sim} g(y')$ an iso in $\mathcal{Y}$ over id_S

- Morphisms: $(x_1, y'_1, h_1) \rightarrow (x_2, y'_2, h_2)$ consists of (α, β) s.t.

$\alpha: x_1 \rightarrow x_2$ & $\beta: y'_1 \rightarrow y'_2$ over some morphism $S_1 \rightarrow S_2$ in $\mathcal{C}$

$$\begin{array}{ccc} f(x_1) & \xrightarrow{f(\alpha)} & f(x_2) \\ \text{s.t. } \downarrow h_1 & & \downarrow h_2 \\ g(y'_1) & \xrightarrow{g(\beta)} & g(y'_2) \end{array} \quad \text{commutes.}$$

Also denote projections $p_1: \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \rightarrow \mathcal{X}$ $((x, y', h) \mapsto x)$

$p_2: \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \rightarrow \mathcal{Y}'$ $((x, y', h) \mapsto y')$

and a 2-isomorphism $\alpha: f \circ p_1 \xrightarrow{\sim} g \circ p_2$ $(\alpha_{(x, y', h)} = f(x) \xrightarrow{h} g(y'))$.

We have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' & \xrightarrow{p_2} & \mathcal{Y}' \\ \downarrow p_1 & \xrightarrow{\sim} & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

Theorem. (universal property)

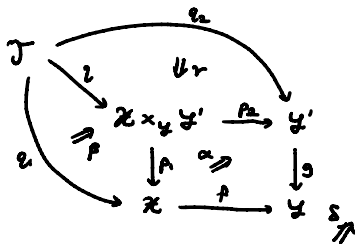
For every 2-commutative diagram of prestacks over $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{q_2} & \mathcal{Y}' \\ \downarrow q_1 & \xrightarrow{\sim} & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

there exists a morphism $\ell: \mathcal{T} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms

$\beta: q_1 \xrightarrow{\sim} p_1 \circ \ell$ and $\gamma: q_2 \xrightarrow{\sim} p_2 \circ \ell$ s.t. [cont'd]

[cont'd] the following diagram is 2-commutative:



that is,

$$\begin{array}{ccc} f \circ e_1 & \xrightarrow{\beta(p)} & f \circ p_1 \circ l \\ \downarrow \delta & & \downarrow \alpha \circ l \\ g \circ e_2 & \xrightarrow{\gamma(r)} & g \circ p_2 \circ l \end{array}$$

commutes.

Moreover, the data (l, β, r) is unique up to unique isomorphism.

(i.e. if $(l, \beta, r), (l', \beta', r')$ satisfy condition,

$$\exists! \text{ 2-isomorphism } \phi: l \xrightarrow{\sim} l' \text{ s.t. } \beta' = p_1(l) \circ \beta, \quad r' = p_2(l) \circ r)$$

Remark. (1) In language of 2-categories, we say $X \times_Y Y'$ is a 2-fiber product.

(2) There is a unique morphism $l: J \rightarrow X \times_Y Y'$ that satisfies the condition w/ $\beta = \text{id}, r = \text{id}$. (In particular, triangles strictly commute)

Def. A 2-commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{e_2} & Y' \\ \downarrow e_1 & \Downarrow \delta & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian if it satisfies the universal property above.

Exercise. (1) The diagram is cartesian iff the induced morphism $l: J \rightarrow X \times_Y Y'$ is iso.

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow & \Downarrow & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow & \Downarrow & \downarrow \\ X & \longrightarrow & Y \end{array} \quad \Downarrow$$

In the 2-commutative diagram, suppose the lower square is cartesian.

Then upper square is cartesian $\Leftrightarrow$ big square is cartesian.

Remark. (1) $\mathcal{C} \xrightarrow{\text{id}} \mathcal{C}$ is an initial object in the 2-category of prestacks over $\mathcal{C}$.

(Moreover, the only 2-morphisms between morphisms with target $\mathcal{C}$ are identities)

So we can define products of prestacks as fiber products over $\mathcal{C}$.

(2) Prestacks over the trivial category are precisely groupoids. Our construction recovers the fiber product of groupoids.

Prop. $\forall S \in \text{Obj}(\mathcal{C}) : (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(S) = \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$
(fiber product of groupoids)

Examples of fiber products of prestacks

(1)
$$\begin{array}{ccc} F' & \longrightarrow & G' \\ \downarrow \square \downarrow & & \\ F & \longrightarrow & G \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{X}_{F'} & \longrightarrow & \mathcal{X}_{G'} \\ \downarrow \square \downarrow & & \\ \mathcal{X}_F & \longrightarrow & \mathcal{X}_G \end{array} \quad \xRightarrow{\text{id}}$$

(fiber product of presheaves)

(2) (magic square) $\mathcal{X}$ prestack over $\mathcal{C}$, $S, T \in \text{Obj}(\mathcal{C})$, $a : S \rightarrow \mathcal{X}$, $b : T \rightarrow \mathcal{X}$.

$$\begin{array}{ccc} S \times_{\mathcal{X}} T & \longrightarrow & S \times T \\ \downarrow \square \downarrow & & \downarrow a \times b \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

(3) (Isom presheaves) $\mathcal{X}$ prestack over $\mathcal{C}$, $S \in \text{Obj}(\mathcal{C})$, $a, b \in \text{Obj}(\mathcal{X}(S))$.

$$\underline{\text{Isom}}_{\mathcal{X}(S)}(a, b) : \mathcal{C}/_S \rightarrow \text{Sets}$$

$$(T \xrightarrow{f} S) \mapsto \text{Mor}_{\mathcal{X}(T)}(f^*a, f^*b)$$

is a presheaf over $\mathcal{C}/_S$.

$$\begin{array}{ccc} \underline{\text{Isom}}_{\mathcal{X}(S)}(a, b) & \longrightarrow & S \\ \downarrow \square \downarrow & & \downarrow (a, b) \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

Note Any prestack over $\mathcal{C}/_S$ is a same thing as a prestack over $\mathcal{C}$ w/ a morphism to S .

(4) (diagrams involving quotient stacks)

$G \rightarrow S$ smooth affine group scheme acting on scheme U over S .

$$(\sigma : G \times_S U \rightarrow U)$$

$$G \times_S U \xrightarrow{\sigma} U$$

$$\downarrow \pi \quad \square \quad \downarrow \rho$$

$$U \xrightarrow{\rho} [U/G]$$

$$G \times_S U \xrightarrow{(\sigma, \rho)} U \times_S U$$

$$\downarrow \quad \square \quad \downarrow \rho \times \rho$$

$$[U/G] \xrightarrow{\Delta} [U/G] \times [U/G]$$

If $P \rightarrow T$ is a principal G -bundle and $P \rightarrow U$ G -equivariant morphism,

$\exists T \rightarrow [U/G]$ unique up to unique isomorphism s.t.

$$P \longrightarrow U$$

$$\downarrow \quad \square \quad \downarrow$$

$$T \longrightarrow [U/G]$$

Stacks

Def. A prestack $\mathcal{X}$ over a site $\mathcal{C}$ is a stack if the following conditions hold for all coverings $\{S_i \rightarrow S\} \in \text{Cov}(S)$:

(1) (morphisms glue) $\forall a, b \in \text{Obj}(\mathcal{X}(S))$ & $\phi_i : a|_{S_i} \rightarrow b$ over $S_i \rightarrow S$ s.t. $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ (pullbacks to morphisms over $S_{ij} = S_i \times_S S_j \rightarrow S$):

$\exists! \phi : a \rightarrow b$ over id_S s.t. $\phi|_{S_i} = \phi_i$.

(2) (objects glue) $\forall a_i \in \text{Obj}(\mathcal{X}(S_i))$ & $\alpha_{ij} : a_i|_{S_{ij}} \xrightarrow{\sim} a_j|_{S_{ij}}$ over $\text{id}_{S_{ij}}$ s.t. $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ (cocycle condition):

$\exists a \in \text{Obj}(\mathcal{X}(S))$ and $\phi_i : a|_{S_i} \xrightarrow{\sim} a_i$ over id_{S_i} s.t. $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

A morphism of stacks is a morphism of prestacks.

Remark. Whether each condition is true is independent of our choice of collection of pullbacks.

Prop. $\mathcal{X}, \mathcal{Y}', \mathcal{Y}$ stacks $\Rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ also a stack.

Theorem. (Stackification) $\mathcal{X}$ a prestack over site $\mathcal{C}$.

$\exists$ stack $\mathcal{X}^{st}$ w/ a morphism $\mathcal{X} \rightarrow \mathcal{X}^{st}$ s.t. for every stack $\mathcal{Y}$,
the induced functor $\text{MOR}(\mathcal{X}^{st}, \mathcal{Y}) \rightarrow \text{MOR}(\mathcal{X}, \mathcal{Y})$ is an equivalence of categories.

Remark. $\mathcal{X}^{st}$ is unique up to isomorphism that is unique up to a unique
2-isomorphism.

Examples of stacks

(1) If F is a presheaf over site $\mathcal{C}$, F is a sheaf iff $\mathcal{X}_F$ is a stack.

On $\text{Sch}_{\text{ét}}$, $\{\text{schemes}\} \subset \{\text{sheaves}\} \subset \{\text{stacks}\}$ (inclusions are fully faithful)
 $\cap$
 $\{\text{presheaves}\} \subset \{\text{prestacks}\}$ & preserves fiber products

(2) Sheaves is a prestack over Sch s.t.

- Objects: (T, F) where T is a scheme and F a $\mathcal{O}_T$ -module
- Morphisms: $(T, F) \rightarrow (T', F')$ consists of $f: T \rightarrow T'$ morphism of schemes and $F' \rightarrow f_* F$ morphism of $\mathcal{O}_{T'}$ -modules s.t. $f^* F' \xrightarrow{\sim} F$.

Sheaves as well as its full subcategories QCoh and Bun are stacks over Sch_{zar} , due to gluing in Zariski topology.

By descent theory, QCoh and Bun are also stacks over $\text{Sch}_{\text{ét}}$, and hence over $\text{Sch}_{\text{ét}}$.

(3) Schemes is a prestack over Sch s.t.

- Objects: $T \rightarrow S$ morphisms of schemes (object over S)
- Morphisms: $(T \rightarrow S) \rightarrow (T' \rightarrow S')$ is a cartesian diagram

$$\begin{array}{ccc} T & \rightarrow & T' \\ \downarrow & \square & \downarrow \\ S & \rightarrow & S' \end{array}$$

This is a stack in Sch_{zar} but not in $\text{Sch}_{\text{ét}}$.

Prop. (moduli stack of smooth curves) If $g \geq 2$, then $\mathcal{M}_g$ is a stack over $\text{Sch}_{\mathbb{Z}}$.

pf Axiom 1 $C \rightarrow S, D \rightarrow S$ families of smooth curves with genus g .

Given $f_i : C_{s_i} \rightarrow D$ (over $S_i \rightarrow S$) s.t. $f_i|_{C_{s_{ij}}} = f_j|_{C_{s_{ij}}}$,
use étale descent for morphisms

$$\begin{array}{ccc} (\sqcup C_{s_i}) \times_c (\sqcup C_{s_i}) & \rightrightarrows & \sqcup C_{s_i} \rightarrow C \\ & \searrow & \downarrow \exists! \\ & & D \end{array}$$

to construct $f : C \rightarrow D$. (= isomorphism also by étale descent)

Axiom 2 Let $C_i \xrightarrow{\pi_i} S_i$ families w/ $\alpha_{ij} : C_i|_{S_{ij}} \xrightarrow{\sim} C_j|_{S_{ij}}$ (over $\text{id}_{S_{ij}}$)

satisfying cocycle condition.

We cannot use effective descent right away on the diagram

$$\begin{array}{ccc} \sqcup C_i & \dashrightarrow & C \quad \exists \text{ (??)} \\ \downarrow & & \downarrow \\ \sqcup S_i & \rightarrow & S \end{array}$$

because we can't ensure $\sqcup C_i \rightarrow \sqcup S_i$ has one of the properties

$\mathcal{P}$ in Prop B.3.1.

Idea Look upon each C_i as a closed subscheme of some projective space bundle $\mathbb{P}(E_i)$, glue E_i to form E , and then use effective descent on $\mathbb{P}(E)$.
 $\mathbb{P}(E_i)$ is a $\mathcal{O}_{S_i}$ -module, E is a $\mathcal{O}_S$ -module.

$$\begin{array}{ccc} \sqcup C_i & \dashrightarrow & C \\ \downarrow & & \downarrow \\ \sqcup \mathbb{P}(E_i) & \rightarrow & \mathbb{P}(E) \end{array}$$

We need the following property of families of smooth curves :

(*) $\pi : C_0 \rightarrow S_0$ family of smooth curves of genus $g \geq 2$.

Then $\omega_{C/S}^{\otimes 3}$ is π -relatively very ample and $\pi_*(\omega_{C/S}^{\otimes 3})$ is a locally free sheaf of rank $5(g-1)$.

In particular, $C_0 \rightarrow \mathbb{P}(\pi_*(\omega_{C/S}^{\otimes 3}))$ which corresponds to $\underbrace{\pi^* \pi_*(\omega_{C/S}^{\otimes 3}) \rightarrow \omega_{C/S}^{\otimes 3}}_{(\text{surj.})}$ is a closed immersion.

(consequence of Prop 5.1.9.)

Set $E_i = (\pi_i)_*(\omega_{C_i/S_i}^{\otimes 3})$ and take closed immersions $C_i \rightarrow \mathbb{P}(E_i)$.

α_{ij} 's induce $\beta_{ij} : E_i|_{S_{ij}} \xrightarrow{\sim} E_j|_{S_{ij}}$ which satisfy cocycle condition.

Descent for quasi-coherent sheaves $\Rightarrow \exists$ quasi-coherent $\mathcal{O}_S$ -module E

& $\psi_i : E|_{S_i} \xrightarrow{\sim} E_i$ s.t. $\beta_{ij} \circ \psi_i|_{S_{ij}} = \psi_j|_{S_{ij}}$.

Again by descent, E is locally free.

$\Rightarrow$ By effective descent (on closed subschemes), we obtain a map

$C \xrightarrow{c_1} \mathbb{P}(E) \rightarrow S$ & $\phi_i : C|_{S_i} \xrightarrow{\sim} C_i$ s.t. $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

$C \rightarrow S$ is smooth and proper b/c these properties are étale-local on the target. $\blacksquare$

Claim. $\underline{\text{Coh}}_{r,d}(C)$, $\underline{\text{Bun}}_{r,d}(C)$, $[U/G]$ are stacks as well.

$\downarrow$ $\downarrow$ $\downarrow$
 over $(\text{Sch}/k)_{\text{ét}}$ over $(\text{Sch}/S)_{\text{ét}}$

(straightforwardly follows from respective descent properties)

Moreover, the canonical fully faithful functor $[U/G]^{\text{pre}} \rightarrow [U/G]$

is a stackification.